Random Walk and Central Limiting Theorem

モデル化とシミュレーション特論
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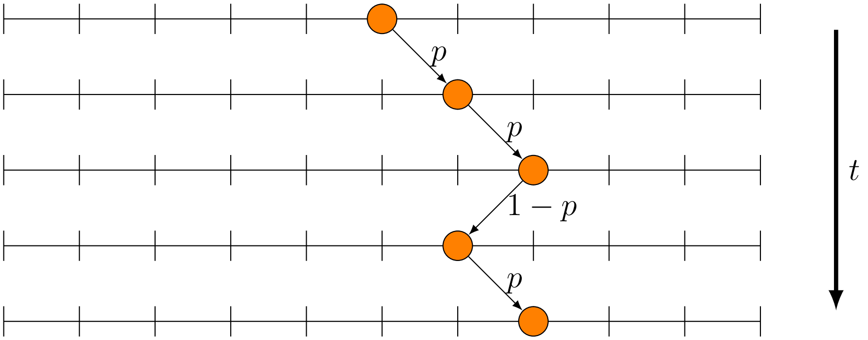
Stochastic Processes (確率過程)

- System evolving non-deterministically
  - evolving with probability
- Random walks (酔歩)
  - Fundamental model of stochastic processes
  - Simplest: one-dimensional lattice
    Move right with p and left with $1-p$ at every step

Sample Program

`https://github.com/modeling-and-simulation-mc-saga/RandomWalk`

Image of random walk



Theoretical Analysis

- Position x of a particle starting from $x = 0$
- At t, a particle position is x if the particle moves right $m = (t + x)/2$ times.
  - Note possible combination of left and right movements.
- Binomial distribution

$$P(x) = \binom{t}{\frac{t+x}{2}} p^{(t+x)/2} (1 - p)^{(t-x)/2} \tag{2.1}$$

Generating Function

- Tool for evaluating moments such as average and deviation.
- Convert probability $P(x)$ for x to probability $Q(m)$ for m

$$P(x) : x = 2m - t, \ \ m \in [0, t] \tag{2.2}$$

$$Q(m) : m = \frac{t + x}{2}, \ \ m \in [0, t] \tag{2.3}$$

- Generating function

$$G(z) = \sum_{m=0}^{t} Q(m) z^m \tag{2.4}$$

General theory of Generating Functions and Moments

$$G(1) = \sum_{m=0}^{t} Q(m) = 1 \tag{2.5}$$

$$G'(z) = \sum_{m=1}^{t} mQ(m)z^{m-1} = \sum_{m=0}^{t} mQ(m)z^{m-1} \tag{2.6}$$

$$G'(1) = \sum_{m=0}^{t} mQ(m) = \langle x \rangle \tag{2.7}$$

$$G''(z) = \sum_{m=2}^{t} m(m-1)Q(m)z^{m-2} = \sum_{m=0}^{t} m(m-1)Q(m)z^{m-2} \tag{2.8}$$

Generating Function for Binomial Distribution

$$G(z) = \sum_{m=0}^{t} \binom{t}{m} p^m (1-p)^{t-m} z^m = (zp + 1 - p)^t \tag{2.10}$$

$$G(1) = 1 \tag{2.11}$$

$$G'(z) = tp\,(zp + 1 - p)^{t-1} \tag{2.12}$$

$$\langle m \rangle = tp \tag{2.13}$$

$$\langle x \rangle = \langle 2m - 1 \rangle = 2tp - t = t\,(2p - 1) \tag{2.14}$$

$$G''(z) = t(t-1)p^2(zp+1-p)^{t-2} \tag{2.15}$$

$$G''(1) = \langle m^2 \rangle - \langle m \rangle = t(t-1)p^2 \tag{2.16}$$

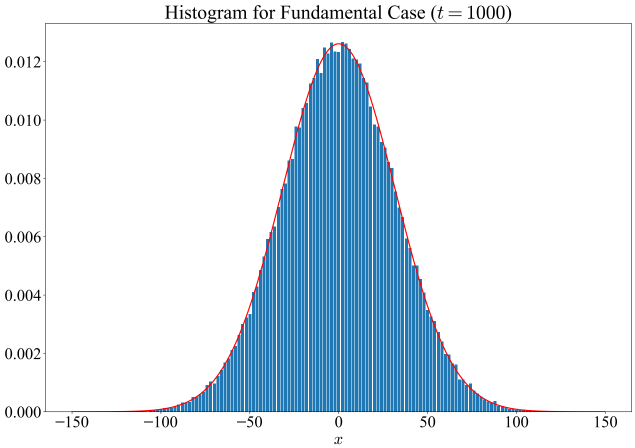
$$\begin{aligned}
\sigma^2 &= \langle x^2 \rangle - \langle x \rangle^2 = \langle 4m^2 - 4mt + t^2 \rangle - \langle x \rangle^2 \\
&= 4\left(\langle m^2 \rangle - \langle m \rangle\right) + 4\langle m \rangle (1-t) + t^2 - \langle x \rangle^2 \\
&= 4G''(1) + 4\langle m \rangle (1-t) + t^2 - \langle x \rangle^2 \\
&= 4tp^2(t-1) + 4tp(1-t) + t^2 - t^2\left(4p^2 - 4p + 1\right) \\
&= \boxed{4tp(1-p)}
\end{aligned} \tag{2.17}$$

Approximated shape at the visinity of the mean

$$P(x) \propto \exp\left[-\frac{(x - \langle x \rangle)^2}{2\sigma^2}\right]$$
$$\langle x \rangle = t\,(2p - 1)$$
$$\sigma^2 = 4tp\,(1 - p)$$

- Normal distribution
- Is this a special for this random walk?

$p = 1/2$ case


Histogram for Fundamental Case ($t = 1000$)
0.000
0.002
0.004
0.006
0.008
0.010
0.012
−150
−100
−50
0
50
100
150
x

Random Walk in other viewpoints

- Sequence of integer random variables $\{X_i\}$ with $P(x)$

$$P(x) = \begin{cases} p & x = 1 \\ 1 - p & x = -1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

- Sum S_i of the random variables

$$S_0 = 0 \tag{3.2}$$

$$S_n = S_{n-1} + X_n = \sum_{k=1}^{n} X_k \tag{3.3}$$

Random walk and sum of random numbers

- S_n: position of a walker at $t = n$
- Distribution of S_n is the distribution of walkers at $t = n$

Extension of one-dimensional Random Walks

- Sequence of random variables $\{X_k\}$ obeying probability density $f(x)$ (x is a continuous random variable)

$$S_0 = 0 \tag{4.1}$$

$$S_n = S_{n-1} + X_n = \sum_{k=1}^{n} X_k \tag{4.2}$$

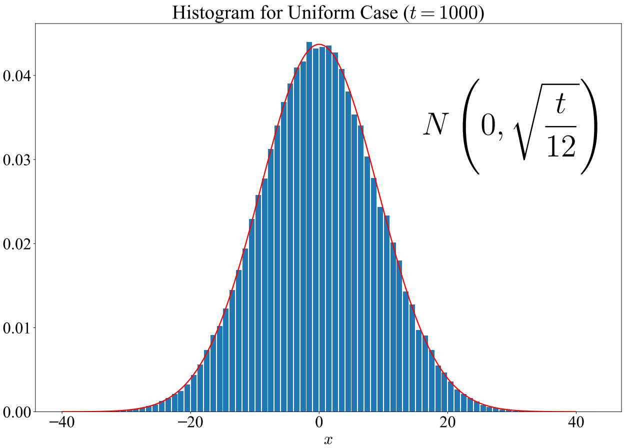
- Sum of independently and identically distributed (i.i.d) random numbers.
- Note that S_i is continuous.

Example: uniform distribution

$$f(x) = \begin{cases} 1 & -\frac{1}{2} \le x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

$$\begin{aligned} \langle x \rangle &= \int_{-1/2}^{1/2} x f(x) \mathrm{d}x = \int_{-1/2}^{1/2} x \mathrm{d}x = \left[\frac{1}{2} x^2 \right]_{-1/2}^{1/2} = 0 \\ \langle x^2 \rangle &= \int_{-1/2}^{1/2} x^2 f(x) \mathrm{d}x = \int_{-1/2}^{1/2} x^2 \mathrm{d}x = \left[\frac{1}{3} x^3 \right]_{-1/2}^{1/2} = \frac{1}{12} \\ \sigma^2 &= \frac{1}{12} \end{aligned} \tag{4.4}$$

Sum of uniform random variables



Example: Exponential distribution

$$f(x) = Ae^{-x}, \quad (0 \leq x < 1) \tag{4.5}$$

$$A = \frac{e}{e-1} \tag{4.6}$$

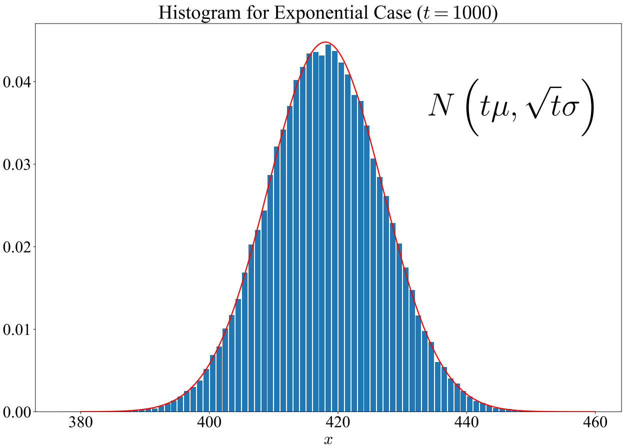
Mean and standard deviation

$$\begin{aligned}\langle x\rangle &= \int_0^1 Axe^{-x}\mathrm{d}x = \int_0^1 Ae^{-x}\mathrm{d}x - A\left[xe^{-x}\right]_0^1 \\ &= 1 - \frac{e}{e-1}e^{-1} = \frac{e-2}{e-1} = 1 - \frac{1}{e-1} \end{aligned} \tag{4.7}$$

$$\begin{aligned}\left\langle x^2\right\rangle &= \int_0^1 Ax^2e^{-x}\mathrm{d}x = 2\int_0^1 Axe^{-x}\mathrm{d}x - A\left[x^2e^{-x}\right]_0^1 \\ &= 2\frac{e-2}{e-1} - \frac{e}{e-1}e^{-1} = \frac{2e-5}{e-1} = 2 - \frac{3}{e-1} \end{aligned} \tag{4.8}$$

$$\sigma^2 = \left\langle x^2\right\rangle - \langle x\rangle^2 = \frac{e^2-3e+1}{(e-1)^2} = 1 - \frac{e}{(e-1)^2} \tag{4.9}$$

Sum of uniform random variables



Central Limiting Theorem (中心極限定理)

- $\{X_k\}$: random variables obeying an identical distribution with the mean μ and deviation σ^2

$$S_n = \sum_{k=1}^{n} X_k \tag{5.1}$$

$$S_n^* = \frac{S_n - n\mu}{\sqrt{n}\sigma} \tag{5.2}$$

$$\lim_{n\to\infty} P(S_n^* \leq a^*) = \frac{1}{\sqrt{2}} \int_{-\infty}^{a*} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}x \tag{5.3}$$

- In the limit $n \to \infty$, S_n^* obeys the standard normal distribution $N\left(0, 1\right)$

Probability Characteristic Function (特性関数)

- Continuous probability density $f(x)$ defined in $[a, b)$

$$G(t) = \int_a^b f(x) e^{itx} \mathrm{d}x \tag{6.1}$$

$$G(0) = \int_a^b f(x) \mathrm{d}x = 1 \tag{6.2}$$

$$G'(t) = \int_a^b ixf(x) e^{itx} \mathrm{d}x \tag{6.3}$$

$$G'(0) = i \int_a^b xf(x) \mathrm{d}x = i \langle x \rangle \tag{6.4}$$

$$G''(t) = -\int_a^b x^2 f(x) e^{itx} \mathrm{d}x \tag{6.5}$$

$$G''(0) = -\int_a^b x^2 f(x) \mathrm{d}x = -\langle x^2 \rangle \tag{6.6}$$

Mean and standard deviation

$$\langle x \rangle = -iG'(0) \tag{6.7}$$

$$\left\langle x^2 \right\rangle = -G''(0) \tag{6.8}$$

$$\sigma^2 = -G''(0) + G'(0)^2 \tag{6.9}$$

Example: uniform

$$f(x) = \begin{cases} 1 & -\frac{1}{2} \leq x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{6.10}$$

$$\begin{aligned} G(t) &= \int_{-1/2}^{1/2} e^{itx} \mathrm{d}x = \left[\frac{1}{it} e^{itx} \right]_{-1/2}^{1/2} \\ &= \frac{1}{it} \left(e^{it/2} - e^{-it/2} \right) = \frac{2 \sin\left(\frac{t}{2}\right)}{t} \\ &= \frac{2}{t} \left(\frac{t}{2} - \frac{1}{6} \left(\frac{t}{2} \right)^3 + \frac{1}{5!} \left(\frac{t}{2} \right)^5 + O\left(t^7\right) \right) \\ &= 1 - \frac{1}{24} t^2 + \frac{1}{5! 2^4} t^4 + O\left(t^6\right) \end{aligned} \tag{6.11}$$

$$G(0) = 1 \tag{6.12}$$

$$G'(t) = -\frac{1}{12}t + \frac{1}{5!2^3}t^3 + O\left(t^5\right) \tag{6.13}$$

$$G'(0) = 0 \tag{6.14}$$

$$G''(t) = -\frac{1}{12} + \frac{3}{5!2^3}t^2 + O\left(t^4\right) \tag{6.15}$$

$$G''(0) = -\frac{1}{12} \tag{6.16}$$

$$\langle x \rangle = 0 \tag{6.17}$$

$$\left\langle x^2 \right\rangle = \frac{1}{12} \tag{6.18}$$

$$\sigma^2 = \frac{1}{12} \tag{6.19}$$

Example: Exponential

$$f(x) = Ae^{-x}, \quad A = \frac{e}{e-1} \tag{6.20}$$

$$G(t) = \int_0^1 Ae^{-x}e^{itx}\mathrm{d}x = \frac{A}{it-1}\left[e^{(it-1)x}\right]_0^1 = \frac{A}{it-1}\left(e^{it-1}-1\right) \tag{6.21}$$

$$G(0) = -A\left(e^{-1}-1\right) = -\frac{e}{e-1}e^{-1}\left(1-e\right) = 1 \tag{6.22}$$

$$G'(t) = -\frac{iA}{\left(it-1\right)^2}\left(e^{it-1}-1\right) + \frac{iA}{it-1}e^{it-1} \tag{6.23}$$

$$G'(0) = -iA\left(e^{-1}-1\right) - iAe^{-1} = i - i\frac{1}{e-1}$$
$$= i\left(1-\frac{1}{e-1}\right) \tag{6.24}$$

$$G''(t) = -\frac{2A}{(it-1)^3}\left(e^{it-1}-1\right)+\frac{2A}{(it-1)^2}e^{it-1}-\frac{A}{it-1}e^{it-1} \tag{6.25}$$

$$G''(0) = 2A\left(e^{-1}-1\right)+2Ae^{-1}+Ae^{-1} = -2+3\frac{1}{e-1} \tag{6.26}$$

$$\langle x \rangle = -i \times i(1 - \frac{1}{e-1}) = 1 - \frac{1}{e-1} \tag{6.27}$$

$$\left\langle x^2 \right\rangle = 2 - 3\frac{1}{e-1} \tag{6.28}$$

$$\begin{aligned} \sigma^2 &= 2 - 3\frac{1}{e-1} - \left(1 - \frac{1}{e-1}\right)^2 \\ &= 2 - \frac{3}{e-1} - 1 + \frac{2}{e-1} - \frac{1}{(e-1)^2} \\ &= 1 - \frac{1}{e-1} - \frac{1}{(e-1)^2} = 1 - \frac{e}{(e-1)^2} \end{aligned} \tag{6.29}$$