Differential Equations: Interacting Oscillators

モデル化とシミュレーション特論
2023 年度前期
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Harmonic oscillator

- Ordinary expression

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \tag{1.1}$$

- Hamiltonian (energy): $p = mv$: momentum

$$H = \frac{p^2}{2m} + V \tag{1.2}$$

$$V = \frac{1}{2}kx^2 \tag{1.3}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} = -kx \tag{1.4}$$

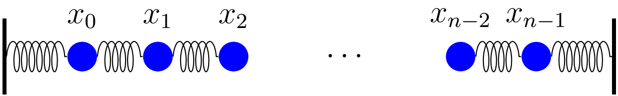
$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -kx$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{1.5}$$

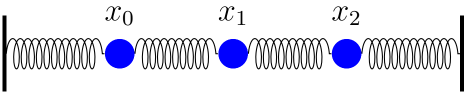
$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

Coupled Harmonic Oscillators: Model

- Connecting n particles of mass m with springs of natural length b.
  - x_i: position of i-th particle
  - $y_i = x_i - (i+1)b$: displacement from resting position of i-th particle



Three particles: Potential energy



Potential energy for each spring

$$\frac{ky_0^2}{2} \qquad \text{0th}$$

$$\frac{k(y_1 - y_0)^2}{2} \qquad \text{1st}$$

$$\frac{k(y_2 - y_1)^2}{2} \qquad \text{2nd}$$

$$\frac{ky_2^2}{2} \qquad \text{3rd}$$

$$V = \frac{k}{2}\left[y_0^2 + \sum_{i=1}^{2}(y_i - y_{i-1})^2 + y_2^2\right] \tag{2.1}$$

Three particles: Equation of motion

$$m\frac{\mathrm{d}^2 y_i}{\mathrm{d}t^2} = -\frac{\partial V}{\partial y_i}, \qquad V = \frac{k}{2}\left[y_0^2 + \sum_{i=1}^{2}\left(y_i - y_{i-1}\right)^2 + y_2^2\right] \tag{2.2}$$

$$\Downarrow$$

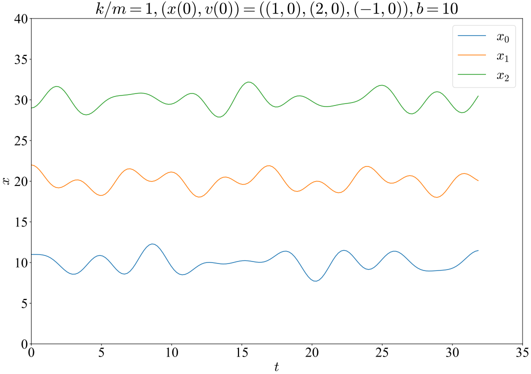
$$m\frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} = -k\left(2y_0 - y_1\right) \tag{2.3}$$

$$m\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} = -k\left(-y_0 + 2y_1 - y_2\right) \tag{2.4}$$

$$m\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2} = -k\left(-y_1 + 2y_2\right) \tag{2.5}$$

Three variables y_i are linearly coupled.

Numerical solutions



Do motions of particles look complicated?

Equation of motion in matrix form

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = -k\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = -kM\vec{y} \tag{3.1}$$

- λ: eigen values of M
- $\vec{v}_\lambda$: eigen vector belonging to λ

$$M\vec{v}_\lambda = \lambda\vec{v}_\lambda \tag{3.2}$$

Eigen values of M

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)^2 - 2\right] = 0 \tag{3.3}$$

$$\lambda_0 = 2 \tag{3.4}$$

$$\lambda_\pm = 2 \pm \sqrt{2} \tag{3.5}$$

Orthnormal eigen vectors of M

$$\lambda_0 = 2$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{v}_0 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.6}$$

$$\lambda_\pm = 2 \pm \sqrt{2}$$

$$\begin{pmatrix} \mp\sqrt{2} & -1 & 0 \\ -1 & \mp\sqrt{2} & -1 \\ 0 & -1 & \mp\sqrt{2} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{v}_\pm = \frac{1}{2} \begin{pmatrix} 1 \\ \mp\sqrt{2} \\ 1 \end{pmatrix} \tag{3.7}$$

Solutions with eigen vectors

Express $\vec{y}$ in linear combination of eigen vectors. Coefficients z_λ are functions of t.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = z_+\vec{v}_+ + z_0\vec{v}_0 + z_1\vec{v}_- \tag{3.8}$$

$$z_\lambda = \vec{v}_\lambda \cdot \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \tag{3.9}$$

$$z_0 = \frac{\sqrt{2}}{2}(y_0 - y_2) \tag{3.10}$$

$$z_\pm = \frac{1}{2}\left(y_0 \mp \sqrt{2}y_1 + y_2\right) \tag{3.11}$$

Equation of motion with eiven vectors

- LHS

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(z_+\vec{v}_+ + z_0\vec{v}_0 + z_-\vec{v}_-\right)$$
$$= m\frac{\mathrm{d}^2 z_+}{\mathrm{d}t^2}\vec{v}_+ + m\frac{\mathrm{d}^2 z_0}{\mathrm{d}t^2}\vec{v}_0 + m\frac{\mathrm{d}^2 z_-}{\mathrm{d}t^2}\vec{v}_-$$

- RHS

$$-kM\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = -kM\left(z_+\vec{v}_+ + z_0\vec{v}_0 + z_-\vec{v}_-\right)$$
$$= -k\left(\lambda_+ z_+\vec{v}_+ + \lambda_0 z_0\vec{v}_0 + \lambda_- z_-\vec{v}_-\right)$$

Equation of motion for each mode

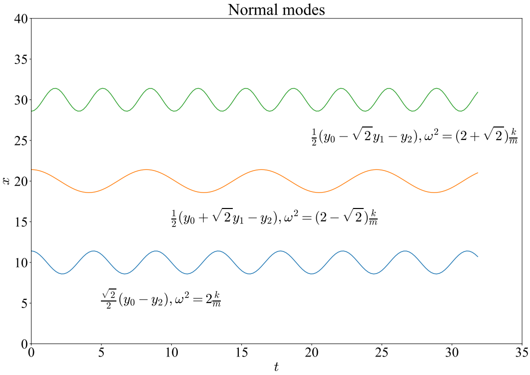
- harmonic oscillator with $\omega_\lambda = \sqrt{k\lambda/m}$

$$m\frac{\mathrm{d}^2 z_\lambda}{\mathrm{d}t^2} = -k\lambda z_\lambda$$

- oscillators with three different angular velocity

$$m\frac{\mathrm{d}^2 z_0}{\mathrm{d}t^2} = -2kz_0$$
$$m\frac{\mathrm{d}^2 z_\pm}{\mathrm{d}t^2} = -k(2 \pm \sqrt{2})z_\pm$$

Motions of eigen modes



Solutions with eigen modes

$$y_0 = \frac{1}{2}\left(z_+ + \sqrt{2}z_0 + z_-\right)$$
$$y_1 = -\frac{\sqrt{2}}{2}\left(z_+ - z_-\right)$$
$$y_2 = \frac{1}{2}\left(z_+ - \sqrt{2}z_0 + z_-\right)$$

Constructor in Java code

```
        equation = (double xx, double[] yy) -> {
            double dy[] = new double[2 * numOscillators];
            //0-th particle
            {
                int i = 0;
                int j = 2 * i;
                dy[j] = yy[j + 1];
                dy[j + 1] = -k * (2 * yy[j] - yy[j + 2]);
            }
            //particles between 1st to n-2-th
            for (int i = 1; i < numOscillators - 1; i++) {
                int j = 2 * i;
                dy[j] = yy[j + 1];
                dy[j + 1] = -k * (-yy[j - 2] + 2 * yy[j] - yy[j + 2]);
            }
            //n-1-th particle
            {
                int i = numOscillators - 1;
                int j = 2 * i;
                dy[j] = yy[j + 1];
                dy[j + 1] = -k * (-yy[j - 2] + 2 * yy[j]);
            }
            return dy;
        };
```

Synchronization

- fire flies
  https://www.youtube.com/watch?v=WMIXp8H8364
- metronomes
  https://www.youtube.com/watch?v=JWToUATLGzs
- pendulum clocks
  Found occasionally by Christiaan Huygens in 1665

Kuramoto Model

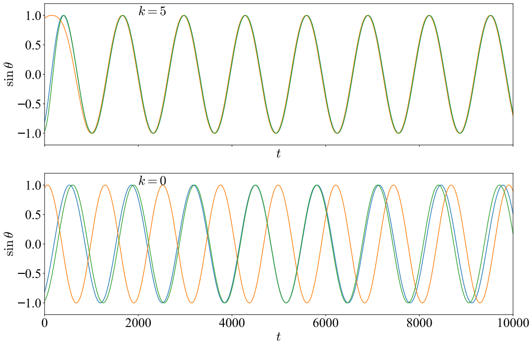
- Fundamental model for synchronization.
- N oscillators interact through their phase differences.

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{k}{N} \sum_j \sin\left(\theta_j - \theta_i\right) \tag{4.1}$$

Description in Kuramoto.java

```
equation = (double tt, double yy[]) -> {
    double dy[] = new double[n];
    for (int i = 0; i < n; i++) {
        dy[i] = omega[i];
        for (int j = 0; j < n; j++) {
            dy[i] += (k / n) * Math.sin(yy[j] - yy[i]);
        }
    }
    return dy;
};
```

Three oscillators



- Not Synchronize with $k = 0$
- Synchronize with $k = 5$

Order Parameter

$$R = \frac{1}{N} \sum_i e^{i\theta_i} \tag{4.2}$$